

REAL NODAL SEXTICS WITHOUT REAL NODES

JOHANNES JOSI

ABSTRACT. We present a rigid isotopy classification of irreducible sextic curves in \mathbb{RP}^2 which have non-real ordinary double points as their only singularities. Our approach uses periods of K3 surfaces and V. Nikulin's classification of "involutions with condition" on unimodular lattices. The classification obtained generalizes Nikulin's rigid isotopy classification of non-singular sextics in \mathbb{RP}^2 .

INTRODUCTION

In 1979, V. Nikulin [14] classified real non-singular sextics (curves of degree six) in \mathbb{RP}^2 up to rigid isotopies, that is, up to deformations in the space of real non-singular sextics. The aim of this article is to generalize this classification to real irreducible sextics whose only singularities are non-real ordinary double points. For such sextics, rigid isotopy means deformation within the space of real nodal sextics with a fixed number of nodes.

Statement of the results. We associate the following invariants with a real irreducible sextic $C \subset \mathbb{CP}^2$ whose only singularities are m pairs of non-real nodes:

- *Isotopy type of the real part.* Since all the nodes are non-real, the real part $\mathbb{RC} \subset \mathbb{RP}^2$ is a collection of smoothly embedded circles, called *ovals*. The way these ovals are disposed in \mathbb{RP}^2 is invariant under rigid isotopies.
- *Dividing type.* Recall that a real non-singular curve C is called *dividing* or *of dividing type I*¹ if $C \setminus \mathbb{RC}$ consists of two connected components, and *of dividing type II* if $C \setminus \mathbb{RC}$ is connected. We extend this notion to nodal curves by declaring that a nodal curve C is dividing if its normalization \tilde{C} is dividing with respect to the inherited real structure. If C is a non-singular dividing curve, then the two connected components of $C \setminus \mathbb{RC}$ are called *halves* of the curve.
- *Number of crossing pairs* (only defined for dividing curves). A non-real node of a dividing curve is called *crossing* if the two branches intersecting at the node belong to different halves of the normalization of the curve. A non-real node is crossing if and only if this is the case for its complex conjugate node, so that we can speak about *crossing* and *non-crossing pairs* of non-real nodes.

These three characteristics are invariant under rigid isotopies. Our first theorem states that they determine the rigid isotopy class.

Theorem 1. *Two real irreducible sextics with m pairs of non-real nodes are rigidly isotopic if and only if their real parts are isotopic, they are of the same dividing type and, if they are dividing, they have the same number of crossing pairs.*

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¹In the literature this is often simply called the *type* of a curve, but this might lead to confusion here since we also use other notions of type, such as the homological type of a curve.

By comparison, Nikulin's theorem states that non-singular real sextics are classified up to rigid isotopy by their isotopy type and their dividing type.

The second question we consider is which combinations of isotopy type, dividing type and number of crossing pairs are realized by real irreducible sextics C with m pairs of non-real nodes. There are several geometric conditions which must be satisfied.

- (1) *Existence of the smoothing.* Smoothing the non-real nodes leads to a non-singular real sextic whose real part has the same isotopy type as the original curve. The smoothed curve is dividing if and only if the original curve is dividing and without crossing pairs. Therefore, a sextic with prescribed isotopy type, dividing type and number of crossing nodes can only exist if there is a non-singular sextic with the same isotopy type and the appropriate dividing type.
- (2) *Harnack's inequality for \tilde{C} .* The normalization \tilde{C} is a smooth real curve of genus $g = 10 - 2m$. Harnack's inequality (see Harnack [5] and Klein [8, p. 154] for a topological proof) for \tilde{C} states that $l \leq g + 1 = 11 - 2m$, where l denotes the number of ovals of $\mathbb{R}C$. Moreover, the equality $l = 11 - 2m$ is only possible if C is dividing.

If C is dividing, then there are further restrictions coming from Rokhlin's complex orientation formula (see [17]). The orientation of each of the halves of \tilde{C} induces a boundary orientation on the real part $\mathbb{R}C$. A pair of ovals in $\mathbb{R}C$ is called *injective* if one oval is contained in the disk bounded by the other oval. An injective pair is called *positive* if the orientation of the two ovals is induced by an orientation of the annulus bounded by these ovals, and *negative* otherwise. Let Π_+ and Π_- denote the numbers of positive and negative injective pairs, respectively. In our case, Rokhlin's complex orientation formula states that

$$2(\Pi_- - \Pi_+) + l = 9 - 2r,$$

where l is the number of ovals and r is the number of crossing pairs. This implies the following relations between the topology of the real part and the number r of crossing pairs for dividing curves:

- (3) *Arnold's congruence.* An oval is called *even* if it lies inside an even number of other ovals, and *odd* otherwise. Arnold's congruence states that

$$o_{\text{even}} - o_{\text{odd}} \equiv 9 - 2r \pmod{4},$$

where o_{even} and o_{odd} denote the numbers of even and odd ovals, respectively.

- (4) *Restriction for curves without injective pairs.* For curves without injective pairs, the complex orientation formula implies

$$l = 9 - 2r.$$

Our second theorem states that these conditions are sufficient for the existence of a real sextic of prescribed isotopy type, dividing type and number of crossing pairs.

Theorem 2. *A triple consisting of an isotopy type, a dividing type and a number of crossing pairs (if the dividing type is I) is realized by a real irreducible sextic whose only singularities are m pairs of non-real nodes if and only if it satisfies the conditions (1)–(4) above.*

Corollary. *The rigid isotopy classes of real irreducible nodal sextics without real nodes are those listed in Figures 1 and 2. In total, there are 78 classes of dividing sextics and 125 classes of non-dividing sextics.*

To describe the isotopy type of the real part, also called real scheme, we use Viro's notation: the symbol $\langle n \rangle$ stands for a collection of n empty ovals; $\langle 1 \langle A \rangle \rangle$ is obtained from A by adding an oval containing all of ovals A in its interior; if A and B are collections of ovals, then $\langle A \sqcup B \rangle$ denotes the disjoint union of A and B , such that no oval of A lies in the interior of an oval of B and vice versa.

Organization of the paper. In Section 1, we review the construction of the K3 surface obtained as a double covering of \mathbb{CP}^2 ramified along a sextic curve $C \subset \mathbb{CP}^2$, first in the complex case and then in the real case. This allows us to associate certain homological data, called the *homological type*, with each real nodal sextic. We introduce the period map for (suitably marked, polarized) real K3 surfaces and prove that rigid isotopy classes of real sextics of a fixed homological type are in bijection with connected components of the corresponding period space, modulo the action of the orthogonal group of the homological type (Proposition 1.18).

In Section 2 we show that, for real nodal irreducible sextics without real nodes, the aforementioned orthogonal group acts transitively on the connected components of the corresponding period space (Proposition 2.1). Therefore, such sextics are rigidly isotopic if and only if they have the same homological type.

In Section 3 we establish a correspondence between certain topological properties of sextics on one hand and arithmetical properties of their homological types on the other hand. In particular, we show how one can tell from the homological type whether a sextic is dividing or not, and we study how the homological type changes when a pair of non-real nodes is perturbed.

Section 4, which is the most technical part of the paper, is devoted to the classification of the homological types of real irreducible sextics without real nodes. We define a set of numerical invariants which completely determines these homological types (Theorem 4.3), and we give necessary and sufficient conditions for the existence of a homological type in terms of these invariants (Theorem 4.4). The proof of these two theorems is based on Nikulin's classification of "involutions with condition" [15].

Finally, in Section 5, we establish a complete dictionary between the topological invariants mentioned in the introduction and the arithmetical invariants introduced in Section 4. This allows us to prove Theorems 1 and 2.

1. NODAL SEXTICS, K3 SURFACES AND PERIODS

In this section we review some facts about K3 surfaces and their periods, and in particular the bijection between rigid isotopy classes of real sextics and chambers of the corresponding real period space, up to the action of a discrete group. For details about K3 surfaces and their periods in general, we refer to the book [2] by W. Barth, K. Hulek, C. Peters and A. van der Ven. For details about "weakly polarized" K3 surfaces, see D. Morrison's article [12]. The case of K3 surfaces obtained from complex nodal sextics is also treated in D. Morrison and M. Saito's article [13]. We borrow most of the notation from A. Degtyarev [3]. The passage from the complex to the real case is done in analogy with similar situations treated by A. Degtyarev, I. Itenberg, V. Kharlamov and V. Nikulin [4, 6, 7, 14].

1.1. Complex nodal sextics and their periods.

The weakly polarized K3 surface obtained from a nodal sextic. A K3 surface is a non-singular, compact complex surface X which is simply connected and has trivial canonical bundle. Endowed with the intersection pairing, which we denote by $x \cdot y$,

the cohomology group $H^2(X; \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$. Note that all even unimodular lattices of signature $(3, 19)$ are isometric.

A *weak polarization* of a K3 surface X is a class $h \in H^2(X, \mathbb{Z})$ of a big and nef line bundle on X , that is, a class h such that $h^2 > 0$ and $h \cdot D \geq 0$ for all irreducible curves $D \subset X$. A *weakly polarized K3 surface* is a pair (X, h) formed by a K3 surface X and a weak polarization h of X . The *nodal classes* of a weakly polarized

m_{\max}									
0	$\langle 1 \sqcup 1 \langle 9 \rangle \rangle$			$\langle 5 \sqcup 1 \langle 5 \rangle \rangle$				$\langle 9 \sqcup 1 \langle 1 \rangle \rangle$	
	$\{0\}$			$\{0\}$				$\{0\}$	
1	$\langle 1 \langle 8 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 7 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 6 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 6 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 7 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 9 \rangle$
	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$
2		$\langle 1 \langle 6 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 7 \rangle$	
		$\{1\}$	$\{0, 2\}$	$\{1\}$	$\{0, 2\}$	$\{1\}$	$\{0, 2\}$	$\{1\}$	
3			$\langle 1 \langle 4 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 5 \rangle$		
			$\{0, 2\}$	$\{1, 3\}$	$\{0, 2\}$	$\{1, 3\}$	$\{2\}$		
4				$\langle 1 \langle 2 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 3 \rangle$		$\langle 1 \langle 1 \rangle \rangle$	
				$\{1, 3\}$	$\{2, 4\}$	$\{3\}$		$\{0\}$	
5					$\langle 1 \rangle$				
					$\{4\}$				

FIGURE 1. Rigid isotopy classes of dividing real nodal sextics with m pairs of non-real nodes of which r are crossing. The set below each real scheme indicates the possible values for the number of crossing pairs r . The total number of pairs m can take any value between r and the upper bound m_{\max} indicated on the left of each row.

m_{\max}									
0	$\langle 1 \langle 9 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 8 \rangle \rangle$		$\langle 4 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 4 \rangle \rangle$		$\langle 8 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 10 \rangle$	
0		$\langle 1 \langle 8 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 7 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 7 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 9 \rangle$	
1		$\langle 1 \langle 7 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 6 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 6 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 8 \rangle$
1			$\langle 1 \langle 6 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 5 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 5 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 7 \rangle$
2			$\langle 1 \langle 5 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 4 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 4 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 6 \rangle$	
2				$\langle 1 \langle 4 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 3 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 3 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 5 \rangle$	
3				$\langle 1 \langle 3 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 2 \rangle \rangle$	$\langle 2 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 4 \rangle$		
3					$\langle 1 \langle 2 \rangle \rangle$	$\langle 1 \sqcup 1 \langle 1 \rangle \rangle$	$\langle 3 \rangle$		
4						$\langle 1 \langle 1 \rangle \rangle$	$\langle 2 \rangle$		
4							$\langle 1 \rangle$		
5								\emptyset	

FIGURE 2. Rigid isotopy classes of non-dividing real nodal sextics with m pairs of non-real nodes. The number of pairs of non-real nodes m can take any value between 0 and the upper bound m_{\max} indicated on the left of each row.

K3 surface (X, h) are the classes $s \in H^2(X; \mathbb{Z})$ of smooth rational curves on X for which $s \cdot h = 0$. Nodal classes are *roots*, *i.e.* vectors with self-intersection -2 .

Let $C \subset \mathbb{CP}^2$ be a reduced (not necessarily irreducible) nodal sextic. Let $\text{bl} : V \rightarrow \mathbb{CP}^2$ be the blow-up of \mathbb{CP}^2 in the nodes of C , and let $\tilde{C} \subset V$ be the strict transform of C . Let $\eta : X \rightarrow V$ be the double covering of V branched along \tilde{C} , and denote the composition $\text{bl} \circ \eta$ by π . Alternatively, π can be obtained by first taking the (singular) double covering $Y \rightarrow \mathbb{CP}^2$ branched along C and then taking the minimal resolution $X \rightarrow Y$, *i.e.* we have the following commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow \eta & \searrow \pi & \downarrow \\ \tilde{C} \subset V & \xrightarrow{\text{bl}} & \mathbb{CP}^2 \supset C \end{array}$$

The surface X is a K3 surface. Let $h \in H^2(X; \mathbb{Z})$ be the first Chern class of the line bundle $\pi^*(\mathcal{O}_{\mathbb{CP}^2}(1))$. Then (X, h) is a weakly polarized K3 surface with $h^2 = 2$, which we call the *weakly polarized K3 surface obtained from the sextic C* . Let $\sigma \subset H^2(X; \mathbb{Z})$ be its set of nodal classes. The nodal curves are precisely the fibres of π above the nodes of C . Let $\Sigma \subset H^2(X; \mathbb{Z})$ be the sublattice generated by σ , and let $S = \Sigma \oplus \mathbb{Z}h \subset H^2(X; \mathbb{Z})$.

Proposition 1.1 (Morrison and Saito [13, Lemma 4.1]). *The curve C is irreducible if and only if S is a primitive sublattice of $H^2(X; \mathbb{Z})$.*

Remark 1.2. More generally, if \tilde{S} denotes the primitive closure of S in $H^2(X; \mathbb{Z})$, then the quotient \tilde{S}/S is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{k-1}$ where k is the number of irreducible components of C .

The following proposition follows from A. Mayer's results on linear systems on K3 surfaces [10], see also [13, Lemma 4.2] and [3, Proposition 3.3.1].

Proposition 1.3. *Let (X, h) be a weakly polarized K3 surface with $h^2 = 2$, and let $\sigma \subset H^2(X; \mathbb{Z})$ be its set of nodal classes. Then the linear system $|h|$ determines a map $X \rightarrow \mathbb{CP}^2$ which is the minimal resolution of a double covering branched along a nodal sextic $C \subset \mathbb{CP}^2$ if and only if the following conditions hold.*

- (i) *The nodal classes are pairwise orthogonal.*
- (ii) *If $s \in L$ is a root such that $2s \in \Sigma$, then $s \in \Sigma$.*
- (iii) *There is no root $s \in \sigma$ such that $s + h \in 2L$.*

Remark 1.4. If the sublattice $S \subset H^2(X; \mathbb{Z})$ generated by h and σ is primitive, then conditions (ii) and (iii) are satisfied.

Periods. Fix an integer $n \leq 10$, interpreted as the number of nodes of the sextics under consideration. We fix once and for all an even unimodular “model lattice” \mathbf{L} of signature $(3, 19)$. Consider pairs (h, σ) , where $h \in \mathbf{L}$ is a vector of square 2 and $\sigma \subset \mathbf{L}$ is a set of n pairwise orthogonal roots orthogonal to h , such that the sublattice $S \subset \mathbf{L}$ generated by σ and h is primitive. We regard two pairs, say (h, σ) and (h', σ') , as equivalent if there is an auto-isometry of \mathbf{L} taking h to h' and σ to σ' .

Proposition 1.5 ([13, Theorem 3.2]). *All pairs (h, σ) as above are equivalent.*

In view of the above proposition, we fix a pair (h, σ) where $h \in \mathbf{L}$ is a vector of square 2 and $\sigma \subset \mathbf{L}$ is a set of n pairwise orthogonal roots, orthogonal to h , such that the sublattice $S \subset \mathbf{L}$ generated by σ and h is primitive. Let $\Sigma \subset \mathbf{L}$ denote the sublattice generated by σ .

Definition 1.6. An (h, σ) -marking of a K3 surface X is an isometry $\mu : \mathbf{L} \rightarrow H^2(X; \mathbb{Z})$ such that $\mu(h)$ is a weak polarization of X and the set of nodal classes of $(X, \mu(h))$ is $\mu(\sigma)$. An (h, σ) -marked K3 surface is a pair (X, μ) formed by a K3 surface X and an (h, σ) -marking μ of X .

Note that a weakly polarized K3 surface admits an (h, σ) -marking if and only if it can be obtained from an irreducible sextic $C \subset \mathbb{CP}^2$ with n nodes.

The Hodge structure of a K3 surface X is determined by the position of the complex line $H^{2,0}(X) \subset H^2(X; \mathbb{C})$. Every non-zero class $\omega_X \in H^{2,0}(X)$ is represented by a non-vanishing holomorphic 2-form on X . It satisfies $\omega_X^2 = 0$ and $\omega_X \cdot \bar{\omega}_X > 0$. Let us denote by $\mathbf{L}_{\mathbb{C}}$ the complex vector space $\mathbf{L} \otimes_{\mathbb{Z}} \mathbb{C}$. The integral bilinear form defined on \mathbf{L} naturally extends to a complex-valued bilinear form on $\mathbf{L}_{\mathbb{C}}$.

Definition 1.7. The period space $\Omega_{h, \sigma}^0$ for (h, σ) -marked K3 surfaces is defined as follows:

$$\begin{aligned} \Omega_{h, \sigma} &= \{\omega \in \mathbf{L}_{\mathbb{C}} \mid \omega^2 = 0, \omega \cdot \bar{\omega} > 0, \omega \cdot h = 0, \omega \perp \sigma\} / \mathbb{C}^* \subset \mathbb{P}(\mathbf{L}_{\mathbb{C}}) \\ \Omega_{h, \sigma}^0 &= \Omega_{h, \sigma} \setminus \bigcup_{\delta \in \Delta} H_{\delta}, \quad \text{where} \quad H_{\delta} = \{[\omega] \in \Omega_{h, \sigma} \mid \omega \cdot \delta = 0\}, \\ \Delta &= \{\delta \in \mathbf{L} \mid \delta^2 = -2, \delta \cdot h = 0, \delta \notin \Sigma\}. \end{aligned}$$

The period of an (h, σ) -marked K3 surface (X, μ) is defined as $[\mu_*^{-1}(\omega_X)] \in \mathbb{P}(\mathbf{L}_{\mathbb{C}})$ where ω_X is a non-zero class in $H^{2,0}(X)$ and μ_* denotes the extension of μ to a map $\mathbf{L}_{\mathbb{C}} \rightarrow H^2(X, \mathbb{C})$.

Remark 1.8.

- (1) The period space $\Omega_{h, \sigma}^0$ is an open set (in the classical topology) in a quadric of dimension $19 - n$. In particular, it is a non-compact complex manifold.
- (2) The periods of (h, σ) -marked K3 surfaces lie in $\Omega_{h, \sigma}^0 \subset \mathbb{P}(\mathbf{L}_{\mathbb{C}})$.

By a family of complex manifolds we mean a proper surjective submersion $f : \mathcal{X} \rightarrow B$ with connected fibres, where \mathcal{X} and B are connected smooth manifolds, together with a complex structure defined on each fibre of f , varying smoothly along the base. Given some type of complex manifolds (such as curves, K3 surfaces, etc.), by a family of this type we mean a family whose fibres are of the said type.

Definition 1.9. A family of (h, σ) -marked K3 surfaces is a pair (\mathcal{X}, μ) , where $f : \mathcal{X} \rightarrow B$ is a family of K3 surfaces and $\mu : \underline{\mathbf{L}}_B \rightarrow R^2 f_*(\mathbb{Z})$ is a sheaf isomorphism such that $\mu_b : \mathbf{L} \rightarrow H^2(X_b; \mathbb{Z})$ is an (h, σ) -marking of X_b for each $b \in B$.

The next theorem is a special case of an analogous statement for weakly polarized marked K3 surfaces by Morrison [12]. It is based on the Global Torelli Theorem for K3 surfaces by I. Shafarevich and I. Piatetski-Shapiro [16] and on the surjectivity of the period map by V. Kulikov [9]. See also [3, Theorem 3.4.1] and [13, Theorem 4.3].

Theorem 1.10 (Morrison [12]). *The space $\Omega_{h, \sigma}^0$ is a fine moduli space for (h, σ) -marked K3 surfaces.*

This means that isomorphism classes of families of (h, σ) -marked K3 surfaces over B are naturally in bijection with smooth maps from B to $\Omega_{h, \sigma}^0$. In particular, there exists a universal family of (h, σ) -marked K3 surfaces over $\Omega_{h, \sigma}^0$ from which every other family is obtained as a pull-back.

Remark 1.11. The equivalence between (h, σ) -marked K3 surfaces and nodal sextics extends to families of (h, σ) -marked K3 surfaces. More precisely, a family of (h, σ) -marked K3 surfaces (\mathcal{X}, μ) determines a bundle of projective planes $\mathbb{P}(E) \rightarrow B$

and an equisingular family of nodal sextics $\mathcal{C} \subset \mathbb{P}(E)$ over B . Indeed, if (\mathcal{X}, μ) is a family of (h, σ) -marked K3 surfaces, then $\mu_b^{-1}(h)$ is a class of Hodge type $(1, 1)$ on each fibre X_b . By the variational Lefschetz theorem on $(1, 1)$ -classes, there is a line bundle \mathcal{L} on \mathcal{X} such that $\mu_b^{-1}(h)$ is the class of the restriction $\mathcal{L}|_b$ for each $b \in B$. Since $\dim \Gamma(X_b, \mathcal{L}|_b) = 3$ independently of $b \in B$, the direct image sheaf $\pi_* \mathcal{L}$ is locally free and defines a rank 3 vector bundle $E \rightarrow B$. The line bundle \mathcal{L} defines a morphism $\mathcal{X} \rightarrow \mathbb{P}(E)$ which is the minimal resolution of a double covering branched along an equisingular family of nodal sextics $\mathcal{C} \subset \mathbb{P}(E)$ over B .

1.2. Real nodal sextics and their periods.

Real structures. A *real structure* on a complex (algebraic or analytic) variety X is an anti-holomorphic involution $c : X \rightarrow X$. Equivalently, a real structure can be viewed as an isomorphism between X and \bar{X} , where \bar{X} is the complex variety with the same underlying space as X , but opposite complex structure, *i.e.*, $\mathcal{O}_{\bar{X}}(U) := \{\bar{f} \mid f \in \mathcal{O}_X(U)\}$. A *real variety* is a pair (X, c) formed by a complex variety X and a real structure c defined on it. The *real points* of a real variety are those points which are fixed by the real structure. By a real structure on a family of complex manifolds, say $f : \mathcal{X} \rightarrow B$, we mean a smooth involution c on \mathcal{X} such that $f \circ c = f$ and the restriction of c to each fibre is anti-holomorphic.

Real sextics and real K3 surfaces. Suppose that $C \subset \mathbb{CP}^2$ is a real sextic, *i.e.* it is invariant under complex conjugation, or equivalently, defined by a polynomial with real coefficients. Let (X, h) be the weakly polarized K3 surface obtained from C . As in § 1.1, let $\pi : X \rightarrow \mathbb{CP}^2$ be the composition $\text{bl} \circ \eta$, where $\text{bl} : V \rightarrow \mathbb{CP}^2$ denotes the blow-up of \mathbb{CP}^2 in the nodes of C and $\eta : X \rightarrow V$ is the double covering of V branched along the strict transform of C . There are two real structures c_1, c_2 on X such that $\pi \circ c_i = \text{conj} \circ \pi$, where $\text{conj} : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ denotes the complex conjugation. We have $c_2 = \tau \circ c_1 = c_1 \circ \tau$, where $\tau : X \rightarrow X$ is the deck transformation of η .

Let c be one of the two real structures. It induces an involutive isometry c^* on the lattice $H^2(X; \mathbb{Z})$. Note that c maps algebraic curves on X to algebraic curves, but it inverts the orientation. We have $c^*(h) = -h$ and $c^*(\sigma) = -\sigma$, where σ is the set of nodal classes of (X, h) . If $s \in \sigma$ is the class of an exceptional curve over a real node, then we have $c^*(s) = -s$. If s' and s'' are the classes of the exceptional curves over a pair of non-real nodes, then we have $c^*(s') = -s''$ and $c^*(s'') = -s'$. Moreover, c^* is an anti-Hodge isometry on $H^2(X; \mathbb{C})$, *i.e.* it maps forms of type (p, q) to forms of type (q, p) . Therefore, the map $x \mapsto c^*(\bar{x})$ defines a real structure on the complex line $H^{2,0}(X)$. Let $\omega_X \in H^{2,0}$ be a nonzero class such that $c^*(\bar{\omega}_X) = \omega_X$. If we write $\omega_X = \omega_+ + i\omega_-$ with $\omega_{\pm} \in H^2(X, \mathbb{R})$, then the conditions $\omega_X^2 = 0$ and $\omega_X \cdot \bar{\omega}_X > 0$ imply that $\omega_+^2 = \omega_-^2 > 0$ and $c^*(\omega_{\pm}) = \pm \omega_{\pm}$. Therefore the three vectors h, ω_+ and ω_- are pairwise orthogonal and span a three-dimensional positive definite subspace of $H^2(X, \mathbb{R})$. In particular, the c^* -invariant sublattice of $H^2(X; \mathbb{Z})$ has one positive square.

Choice of the real structure. It is not possible to coherently pick one of the two real structures for all real nodal sextics. This is possible however if we consider only real nodal sextics without real nodes. Let c_1 and c_2 be the two real structures, and let $X^{c_1}, X^{c_2} \subset X$ be their respective sets of fixed points. We have $X^{c_i} = \pi^{-1}(B_i)$, where B_1 and B_2 are the two (not necessarily connected) regions delimited by \mathbb{RC} inside \mathbb{RP}^2 ; more precisely, if $f \in \mathbb{R}[x_0, x_1, x_2]$ is a polynomial defining C , then we have $B_1 = \{x \in \mathbb{RP}^2 \mid f(x) \geq 0\}$ and $B_2 = \{x \in \mathbb{RP}^2 \mid f(x) \leq 0\}$ or vice versa. If

C is a sextic with only non-real nodes, then, like in the case of non-singular sextics, exactly one of the two regions B_1 and B_2 is non-orientable. The real structure c whose fixed point set covers the non-orientable region of $\mathbb{RP}^2 \setminus \mathbb{RC}$ is distinguished by the following homological property (see [14, p. 163]):

(*) There is an element $x \in H^2(X; \mathbb{Z})$ with $c^*(x) = -x$ and $x \cdot h \equiv 1 \pmod{2}$.

In the special case where the real part of the curve C is empty, one real structure has no fixed points, and the fixed point set of the other real structure is a non-trivial unramified covering of \mathbb{RP}^2 . In this case the real structure with property (*) is the one without fixed points. For the classification, it is convenient to consistently consider one of the two real structures. In the following, we always suppose that the real structure c with property (*) is chosen. We call the triple (X, h, c) the *weakly polarized real K3 surface obtained from C* .

Real periods. As in § 1.1, we fix a vector $h \in \mathbf{L}$ and a set of nodal classes $\sigma \subset \mathbf{L}$.

Definition 1.12. An involutive isometry ϕ on \mathbf{L} is called *geometric* if $\phi(h) = -h$, $\phi(\sigma) = -\sigma$ and the ϕ -invariant sublattice $\mathbf{L}_+ = \{x \in \mathbf{L} \mid \phi(x) = x\}$ has one positive square.

Definition 1.13. Fix a geometric involution ϕ on \mathbf{L} . An (h, σ, ϕ) -*marking* of a real K3 surface (X, c) is an (h, σ) -marking of X such that $\mu \circ \phi = c^* \circ \mu$. An (h, σ, ϕ) -*marked K3 surface* is a triple (X, c, μ) where (X, c) is a real K3 surface and μ is an (h, σ, ϕ) -marking of (X, c) . A real sextic C is of *homological type* $(\mathbf{L}, h, \sigma, \phi)$ if the real weakly polarized K3 surface obtained from C admits a (h, σ, ϕ) -marking.

Definition 1.14. The *period space* $\Omega_{h, \sigma, \phi}^0$ for (h, σ, ϕ) -marked real K3 surfaces is defined as follows:

$$\begin{aligned} \Omega_{h, \sigma, \phi} &= \{\omega \in \mathbf{L}_{\mathbb{C}} \mid \omega^2 = 0, \omega \cdot \bar{\omega} > 0, \omega \cdot h = 0, \omega \perp \sigma, \phi_*(\omega) = \bar{\omega}\} / \mathbb{R}^* \\ \Omega_{h, \sigma, \phi}^0 &= \Omega_{h, \sigma, \phi} \setminus \bigcup_{\delta \in \Delta} H_{\delta}, \quad \text{where} \quad H_{\delta} = \{[\omega] \in \Omega_{h, \sigma, \phi} \mid \omega \cdot \delta = 0\}, \\ \Delta &= \{\delta \in \mathbf{L} \mid \delta^2 = -2, \delta \cdot h = 0, \delta \notin \Sigma\}. \end{aligned}$$

The *period* of an (h, σ, ϕ) -marked real K3 surface (X, c, μ) is defined as $[\mu_*^{-1}(\omega_X)] \in \Omega_{h, \sigma, \phi}^0$, where ω_X is a non-zero class in $H^{2,0}(X)$ such that $c^*\omega_X = \bar{\omega}_X$, and μ_* denotes the extension of μ to a map $\mathbf{L}_{\mathbb{C}} \rightarrow H^2(X, \mathbb{C})$.

The fact that ϕ is geometric implies that $\Omega_{h, \sigma, \phi}$ is non-empty. The subspaces $H_{\delta} \subset \Omega_{h, \sigma, \phi}$ can be of real codimension one or two depending on δ . Therefore the real period space $\Omega_{h, \sigma, \phi}^0$ is non-empty, but in general not connected.

Definition 1.15. A *family of (h, σ, ϕ) -marked real K3 surfaces* is a triple (\mathcal{X}, μ, c) , where (\mathcal{X}, μ) is a family of (h, σ) -marked K3 surfaces c is and a real structure on \mathcal{X} , such that $\mu_b : \mathbf{L} \rightarrow H^2(X_b; \mathbb{Z})$ is an (h, σ, ϕ) -marking of (X_b, c_b) for each $b \in B$, where c_b denotes the restriction of c to X_b .

If ϕ is a geometric involution, then $x \mapsto \phi(\bar{x})$ defines a real structure on the complex period space $\Omega_{h, \sigma}^0$, and the real period space $\Omega_{h, \sigma, \phi}^0 \subset \Omega_{h, \sigma}^0$ is the set of real points for this real structure.

Lemma 1.16. Let (\mathcal{X}, μ) be a family of (h, σ) -marked sextics whose periods are contained in $\Omega_{h, \sigma, \phi}^0 \subset \Omega_{h, \sigma}^0$. Then there exists a unique real structure c on \mathcal{X} such that (\mathcal{X}, μ, c) is a family of (h, σ, ϕ) -marked real K3 surfaces.

Proof. Consider the family $\overline{\mathcal{X}}$ obtained from \mathcal{X} by inverting the complex structure on each fibre. Then $\mu \circ \phi$ defines an (h, σ) -marking of $\overline{\mathcal{X}}$, and the periods of the marked families (\mathcal{X}, μ) and $(\overline{\mathcal{X}}, \mu \circ \phi)$ coincide. Since $\Omega_{h, \sigma}^0$ is a fine moduli space for (h, σ) -marked sextics, this implies that there is a unique isomorphism $c : \mathcal{X} \rightarrow \overline{\mathcal{X}}$ such that $\mu \circ \phi = c^* \circ \mu$. In other words, c defines a real structure on \mathcal{X} such that (\mathcal{X}, μ, c) is a family of (h, σ, ϕ) -marked real K3 surfaces. \square

Corollary 1.17. *The space $\Omega_{h, \sigma, \phi}^0$ is a fine moduli space for (h, σ, ϕ) -marked real K3 surfaces.*

The orthogonal group of a homological type $(\mathbf{L}, h, \sigma, \phi)$ is

$$\mathrm{O}(\mathbf{L}, h, \sigma, \phi) = \{\psi : \mathbf{L} \rightarrow \mathbf{L} \text{ isometry} \mid \psi(h) = h, \psi(\sigma) = \sigma \text{ and } \psi \circ \phi = \phi \circ \psi\}.$$

Proposition 1.18. *The period map induces a one-to-one correspondence between rigid isotopy classes of real irreducible nodal sextics of homological type $(\mathbf{L}, h, \sigma, \phi)$ and connected components of $\Omega_{h, \sigma, \phi}^0$ modulo the action of $\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$.*

Proof. Consider the space $\mathbb{R}P H^0(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(6))^* \cong \mathbb{RP}^{27}$ parameterizing real plane sextics, and let $W \subset \mathbb{RP}^{27}$ be the subset parameterizing real irreducible nodal sextics $C \subset \mathbb{CP}^2$ of homological type $(\mathbf{L}, h, \sigma, \phi)$. Rigid isotopy classes of such sextics are connected components of W . Let U be the set of pairs (C, μ) where $C \in W$ and μ is an (h, σ, ϕ) -marking of the weakly polarized real K3 surface obtained from C . Let $p_1 : U \rightarrow W$ be the map forgetting the marking.

Consider the universal curve $\mathcal{C}_W = \{(C, x) \in W \times \mathbb{CP}^2 \mid x \in C\}$ over W , and let \mathcal{X}_W be the family of K3 surfaces obtained from $\mathcal{C}_W \subset W \times \mathbb{CP}^2$. The projection $\mathcal{X}_W \rightarrow W$ is topologically a locally trivial fibration. Therefore, once we fix an open contractible subset $N \subset W$, we have natural isometries between $H^2(X_C; \mathbb{Z})$ and $H^2(X_{C'}; \mathbb{Z})$ for $C, C' \in N$, i.e. markings can be extended to nearby sextics. This defines a topology on U such that the projection $p_1 : U \rightarrow W$ is a regular, unramified covering with deck transformation group $\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$. Therefore, p_1 induces a bijection between $\pi_0(U)/\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$ and $\pi_0(W)$.

Let $p_2 : U \rightarrow \Omega_{h, \sigma, \phi}^0$ be the period map. The group $\mathrm{PGL}_3(\mathbb{R})$ naturally acts on U , and the period map is invariant with respect to this action. We claim that $p_2 : U \rightarrow \Omega_{h, \sigma, \phi}^0$ is a $\mathrm{PGL}_3(\mathbb{R})$ -principal bundle. Assuming this claim, p_2 is a locally trivial fibration with connected fibres, so it induces a bijection between $\pi_0(U)$ and $\pi_0(\Omega_{h, \sigma, \phi}^0)$. Moreover, since p_2 is equivariant with respect to the action of $\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$, it gives rise to a bijection between the quotients $\pi_0(U)/\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$ and $\pi_0(\Omega_{h, \sigma, \phi}^0)/\mathrm{O}(\mathbf{L}, h, \sigma, \phi)$, which proves the proposition.

To prove the claim, consider the universal family (\mathcal{X}, μ, c) of real (h, σ, ϕ) -marked K3 surfaces over the real period space $\Omega_{h, \sigma, \phi}^0$. Let $\mathcal{L} \rightarrow \mathcal{X}$ be the line bundle defined by the polarization and let E be the rank 3 vector bundle over $\Omega_{h, \sigma, \phi}^0$ defined by $\pi_* \mathcal{L}$ (cf. Remark 1.11). The real structure c induces a real structure on E ; let $\mathbb{R}E$ denote the real part of E , which is a real vector bundle. The elements $(C, \mu) \in p_2^{-1}([\omega])$ naturally correspond to isomorphisms between $\mathbb{P}(\mathbb{R}E_{[\omega]})$ and a fixed real projective plane. Therefore we can identify U with the projective frame bundle of $\mathbb{R}E$, which clearly is a $\mathrm{PGL}_3(\mathbb{R})$ -principal bundle. \square

2. HOMOLOGICAL QUASI-SIMPLICITY

Our aim in this section is to prove the following proposition.

Proposition 2.1. *For real irreducible nodal sextics without real nodes, the rigid isotopy class is determined by the homological type.*

By Proposition 1.18, the rigid isotopy classes of curves with homological type $(\mathbf{L}, h, \sigma, \phi)$ are in bijection with the connected components of $\Omega_{h, \sigma, \phi}^0$ modulo the action of the group $O(\mathbf{L}, h, \sigma, \phi)$. Hence, to show that there is only one rigid isotopy class for a given homological type, we only need to show that $O(\mathbf{L}, h, \sigma, \phi)$ acts transitively on the connected components of $\Omega_{h, \sigma, \phi}^0$.

Walls in the period space. We define lattices K_+ and K_- by

$$K_{\pm} = \{x \in \mathbf{L} \mid x \cdot h = 0, x \perp \sigma, \phi(x) = \pm x\}.$$

Both K_+ and K_- have one positive square; let \mathbb{H}_+ and \mathbb{H}_- be the hyperbolic spaces associated with them, *i.e.* $\mathbb{H}_{\pm} = \{x \in K_{\pm} \otimes \mathbb{R} \mid x^2 > 0\} / \mathbb{R}^*$. For an element $[\omega] \in \Omega_{h, \sigma, \phi}$, we can write $\omega = \omega_+ + i\omega_-$ with $\omega_+, \omega_- \in \mathbf{L}_{\mathbb{R}}$. The definition of $\Omega_{h, \sigma, \phi}$ implies that $\omega_+ \in K_+ \otimes \mathbb{R}$, $\omega_- \in K_- \otimes \mathbb{R}$ and $\omega_+^2 = \omega_-^2 > 0$. The map $\Omega_{h, \sigma, \phi} \rightarrow \mathbb{H}_+ \times \mathbb{H}_-$ sending $[\omega]$ to the pair $([\omega_+], [\omega_-])$ is a trivial two-sheeted covering. The two connected components of $\Omega_{h, \sigma, \phi}$ are interchanged by the automorphism $-\phi \in O(\mathbf{L}, h, \sigma, \phi)$. Therefore, instead of studying the connected components of $\Omega_{h, \sigma, \phi}^0$, we may equivalently study their images in $\Omega_{h, \sigma, \phi} / \{1, -\phi\} \cong \mathbb{H}_+ \times \mathbb{H}_-$.

Recall that the real period space $\Omega_{h, \sigma, \phi}^0$ is obtained from $\Omega_{h, \sigma, \phi}$ by removing the orthogonal complements H_{δ} of all vectors δ in $\Delta = \{\delta \in \mathbf{L} \mid \delta^2 = -2, \delta \cdot h = 0, \delta \notin \Sigma\}$. Consider such a vector $\delta \in \Delta$. We write it as $\delta = \delta_{\Sigma} + \delta_+ + \delta_-$, where $\delta_{\Sigma} \in \Sigma \otimes \mathbb{Q}$, $\delta_+ \in K_+ \otimes \mathbb{Q}$ and $\delta_- \in K_- \otimes \mathbb{Q}$. An element $[\omega_+ + i\omega_-] \in \Omega_{h, \sigma, \phi}$ is orthogonal to δ if and only if both $\omega_+ \cdot \delta_+ = 0$ and $\omega_- \cdot \delta_- = 0$.

Let $H_{\delta_{\pm}} = \{x \in \mathbb{H}_{\pm} \mid x \cdot \delta_{\pm} = 0\}$. We have

$$H_{\delta_{\pm}} = \begin{cases} \mathbb{H}_{\pm} & \text{if } \delta_{\pm} = 0, \\ \text{a real hyperplane in } \mathbb{H}_{\pm} & \text{if } \delta_{\pm}^2 < 0, \\ \text{empty} & \text{otherwise,} \end{cases}$$

and $H_{\delta} = \{[\omega_+ + i\omega_-] \in \Omega_{h, \sigma, \phi} \mid ([\omega_+], [\omega_-]) \in H_{\delta_+} \times H_{\delta_-} \subset \mathbb{H}_+ \times \mathbb{H}_-\}$. Therefore, the subspace H_{δ} can be either empty or of codimension one or two in $\Omega_{h, \sigma, \phi}$. Since we are only interested in the connected components of the period space, we may disregard the walls of codimension two. The only relevant walls are those defined by a vector δ for which one of δ_+ and δ_- is zero and the other one has a negative square.

Reflections. Let L be a non-degenerate even lattice, and let $v \in L$ be a primitive vector with $v^2 = -2k \neq 0$. Consider the hyperplane $H_v \subset L \otimes \mathbb{R}$ orthogonal to v , and let R_v be the reflection in this hyperplane. It is defined by the formula

$$R_v(x) = x + \frac{x \cdot v}{k} v.$$

The reflection R_v maps the lattice L to itself if and only if $x \cdot v \in k\mathbb{Z}$ for all $x \in L$. In particular, this is always the case if $k = 1$, that is, $v^2 = -2$.

Proof of Proposition 2.1. It is sufficient to show the following: *For each $\delta \in \Delta$ such that $H_{\delta} \subset \Omega_{h, \sigma, \phi}$ is a real hyperplane, there is an automorphism $\rho \in O(\mathbf{L}, h, \sigma, \phi)$ which acts on $\Omega_{h, \sigma, \phi}$ as a reflection in H_{δ} .* Let $\delta = \delta_{\Sigma} + \delta_+ + \delta_- \in \Delta$. As discussed

above, H_δ is a real hyperplane in $\Omega_{h,\sigma,\phi}$ if and only if one of δ_+ and δ_- is zero and the other one has a negative square; let $\varepsilon \in \{\pm 1\}$ be such that $\delta_\varepsilon^2 < 0$ and $\delta_{-\varepsilon} = 0$. Let us write $\delta = \frac{1}{2}(v + w)$ where $v \in \Sigma$ and w is orthogonal to S , with $\phi(w) = \varepsilon w$ and $w^2 < 0$. Since v^2 and w^2 are both non-positive and $v^2 + w^2 = 4\delta^2 = -8$, we have $v^2 \in \{0, -2, -4, -6\}$. Therefore, the vector $v \in \Sigma$ is either zero or a sum of up to three elements of $\pm\sigma$. We label the nodal classes $\sigma = \{s'_1, s''_1, \dots, s'_m, s''_m\}$, such that $\phi(s'_i) = -s''_i$ and $\phi(s''_i) = -s'_i$. Since $\delta - \varepsilon\phi(\delta) = \frac{1}{2}(v - \varepsilon\phi(v)) \in L$, a root s'_i appears in v if and only if s''_i appears as well. This leaves two possibilities: either $v = 0$, or $v = \pm s'_k \pm s''_k$ for some k . In the first case ($v = 0$) we have $\delta = w$, and $\rho = R_\delta$ is an automorphism of $(\mathbf{L}, h, \sigma, \phi)$ which acts on $\Omega_{h,\sigma,\phi}$ by reflection in H_δ . Let us consider the second case, where $v = \pm s'_k \pm s''_k$ for some k . Since changing the signs in front of s'_k and s''_k gives another vector in Δ defining the same hyperplane H_δ , we may assume $v = s'_k + \varepsilon s''_k$. Then δ and $\phi(\delta)$ are orthogonal to each other, and $R_\delta \circ R_{\phi(\delta)} = R_w \circ R_v$ is an automorphism of \mathbf{L} which acts on $\Omega_{h,\sigma}$ by reflection in $H_\delta = H_w$. However, it does not necessarily map the set σ to itself. More precisely, it acts on Σ by reflection in the hyperplane orthogonal to $v = s'_k + \varepsilon s''_k$. To ensure that the set σ is mapped to itself, we compose $R_\delta \circ R_{\phi(\delta)}$ with a rotation in the plane $\langle s'_k, s''_k \rangle$ if $\varepsilon = 1$. Indeed,

$$\rho := \begin{cases} R_{s'_k} \circ R_{s''_k} \circ R_\delta \circ R_{\phi(\delta)} & \text{if } \varepsilon = +1, \\ R_\delta \circ R_{\phi(\delta)} & \text{if } \varepsilon = -1 \end{cases}$$

is an automorphism of $(\mathbf{L}, h, \sigma, \phi)$ which acts on $\Omega_{h,\sigma,\phi}$ by reflection in H_δ . \square

3. DEDUCING TOPOLOGICAL INFORMATION FROM THE HOMOLOGICAL TYPE

In this section we study how the homological type of a real sextic encodes topological information such as the dividing type and the number of crossing pairs.

3.1. Dividing type. Recall that an element α of a non-degenerate lattice L is called *characteristic* if we have $x^2 \equiv \alpha \cdot x \pmod{2}$ for all elements $x \in L$. Characteristic elements always exist, and they are uniquely defined up to $2L$, *i.e.* the residue $\bar{\alpha} \in L/2L$ does not depend on the choice of α . A lattice is even if and only if zero is a characteristic element. Given an involution ϕ on a lattice L , we may consider the *twisted bilinear form* given by $(x, y) \mapsto x \cdot \phi(y)$. We say that an element α is characteristic for the involution ϕ if it is characteristic for the corresponding twisted bilinear form.

Let (X, h) be the weakly polarized K3 surface obtained from a real irreducible nodal sextic $C \subset \mathbb{CP}^2$, and let c_1 and c_2 be the real structures on X which lift the complex conjugation of \mathbb{CP}^2 . Let α_1 and $\alpha_2 \in H^2(X; \mathbb{Z})$ be characteristic elements for the induced involutions c_1^* and c_2^* on $H^2(X; \mathbb{Z})$, and let $\bar{\alpha}_1$ and $\bar{\alpha}_2$ denote their residues in $H^2(X; \mathbb{Z}/2\mathbb{Z})$. The classes $\bar{\alpha}_i$ are Poincaré dual to the $\mathbb{Z}/2\mathbb{Z}$ -fundamental classes of the fixed-point sets X^{c_i} .

Note that if the real part of C is non-empty, then C is dividing if and only if the class of $\mathbb{R}\tilde{C}$ is trivial in $H_1(\tilde{C}; \mathbb{Z}/2\mathbb{Z})$. By Proposition 1.1, the sublattice $S \subset H^2(X; \mathbb{Z})$ is primitive, so that we have a natural embedding $S/2S \hookrightarrow H^2(X; \mathbb{Z}/2\mathbb{Z})$.

Proposition 3.1. *The elements $\bar{\alpha}_1, \bar{\alpha}_2 \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ are contained in the image of $S/2S$ if and only if $[\mathbb{R}\tilde{C}]$ is trivial in $H_1(\tilde{C}; \mathbb{Z}/2\mathbb{Z})$.*

To prove this proposition, we need the following lemma. Recall that V is the blow-up of \mathbb{CP}^2 in the nodes of C and that $\eta : X \rightarrow V$ is the double covering branched along \tilde{C} , with deck transformation τ . The *transfer map* $\text{tr} : H_2(V, \tilde{C}; \mathbb{Z}/2\mathbb{Z}) \rightarrow$

$H_2(X; \mathbb{Z}/2\mathbb{Z})$ is obtained by mapping a relative 2-cell in V to the sum of its two preimages in X .

Lemma 3.2. *The transfer map $\text{tr} : H_2(V, \tilde{C}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_2(X; \mathbb{Z}/2\mathbb{Z})$ is injective.*

Proof. In the following, we use $\mathbb{Z}/2\mathbb{Z}$ coefficients for homology and cohomology unless stated otherwise. Consider the Smith exact homology sequence (see [4, p. 3]) for the pair (X, τ) .

$$\cdots \rightarrow H_3(X) \xrightarrow{\eta^*} H_3(V, \tilde{C}) \xrightarrow{\Delta} H_2(V, \tilde{C}) \oplus H_2(\tilde{C}) \xrightarrow{\text{tr} + i_*} H_2(X) \rightarrow \cdots$$

Here, i_* is the homomorphism induced by the inclusion $\tilde{C} \hookrightarrow X$. The second component of the connecting homomorphism Δ is the boundary map $\partial : H_3(V, \tilde{C}) \rightarrow H_2(\tilde{C})$ which is injective because its kernel coincides with the image of $H_3(V) = 0$. To show that the transfer map is injective, suppose $x \in H_2(V, \tilde{C})$ lies in its kernel. Then $(x, 0)$ lies in the kernel of $\text{tr} + i_*$, which coincides with the image of Δ . Because Δ is injective on the second component, this implies $x = 0$. \square

Proof of Proposition 3.1. Consider the following diagram, where the bottom row is part of the long exact homology sequence for the pair (V, \tilde{C}) :

$$\begin{array}{ccccccc} & & & H_2(X) & & & \\ & & & \uparrow \text{tr} & & & \\ \cdots & \rightarrow & H_2(V) & \xrightarrow{\rho} & H_2(V, \tilde{C}) & \xrightarrow{\partial} & H_1(\tilde{C}) \rightarrow \cdots \end{array}$$

Note that the composition $\text{tr} \circ \rho$ corresponds to $\eta^* : H^2(V) \rightarrow H^2(X)$ if we identify homology and cohomology via Poincaré duality. A basis for $H_2(V)$ is given by the strict transform of a line in \mathbb{CP}^2 and the exceptional classes of the blow-up. Therefore the image of η^* is generated by the mod 2 residues of h and σ , and the image of η^* is $S/2S \subset H^2(X)$. Therefore, $\bar{\alpha}_i$ is contained in $S/2S$ if and only if $[X^{c_i}]$ is contained in $\text{im}(\text{tr} \circ \rho)$. Let B_1 and B_2 be the (not necessarily connected) regions delimited by $\mathbb{R}\tilde{C}$ in \mathbb{RP}^2 . Recall that the fixed point sets X^{c_i} are double coverings of the regions B_i under η , and hence we have $[X^{c_i}] = \text{tr}([B_i])$. Suppose that $[\mathbb{R}\tilde{C}]$ is trivial in $H_1(\tilde{C})$. Since $\partial[B_i] = [\mathbb{R}\tilde{C}] = 0$ in $H_1(\tilde{C})$, the classes $[B_i]$ lie in the image of ρ . Therefore, $[X^{c_i}] = \text{tr}([B_i])$ is contained in $\text{im}(\text{tr} \circ \rho)$ and $\bar{\alpha}_i$ is contained in $S/2S$. Now suppose that $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are contained in $S/2S$. Then the classes $[X^{c_i}]$ lie in the image of $\text{tr} \circ \rho$. Because the transfer map is injective by Lemma 3.2, this implies that the classes $[B_i]$ are in the image of ρ , and therefore $\partial[B_i] = [\mathbb{R}\tilde{C}] = 0$ in $H_1(\tilde{C})$. \square

Remark 3.3. If C is dividing, then $\bar{\alpha}_1$ and $\bar{\alpha}_2$ lie in fact in the c_i^* -invariant part of $S/2S$. If C has only non-real nodes, with classes $s'_1, s''_1, \dots, s'_m, s''_m$, then the c_i^* -invariant part of $S/2S$ is generated by the mod 2 residues of $h, s'_1 + s''_1, \dots, s'_m + s''_m$.

Lemma 3.4. *If C is an irreducible sextic with only non-real nodes, then $\bar{\alpha}_1 + \bar{\alpha}_2 = \bar{h}$.*

Proof. Note that $\mathbb{R}V \subset V$ is homologous to the preimage of a general line in \mathbb{CP}^2 . Indeed, $\mathbb{R}V$ intersects the preimage of a non-real line in one point and does not intersect any of the exceptional curves since all the nodes are non-real. Therefore, $\eta^*([\mathbb{R}V]^{\text{PD}}) = \bar{h}$, where PD denotes Poincaré duality and $\bar{h} \in H^2(X)$ is the mod 2 residue of h . As noted in the proof of Proposition 3.1, the map η^* corresponds to

$\mathrm{tr} \circ \rho$ under Poincare duality. Hence, we have

$$\begin{aligned} \bar{h} &= \eta^*([\mathbb{R}V]^{\mathrm{PD}}) = (\mathrm{tr} \circ \rho[\mathbb{R}V])^{\mathrm{PD}} = (\mathrm{tr}([B_1] + [B_2]))^{\mathrm{PD}} \\ &= \mathrm{tr}([B_1])^{\mathrm{PD}} + \mathrm{tr}([B_2])^{\mathrm{PD}} = [X^{c_1}]^{\mathrm{PD}} + [X^{c_2}]^{\mathrm{PD}} = \bar{\alpha}_1 + \bar{\alpha}_2. \end{aligned} \quad \square$$

Corollary 3.5. *If C is a dividing real irreducible sextic with only non-real nodes, then exactly one of the classes $\bar{\alpha}_1$ and $\bar{\alpha}_2$ is contained in $\Sigma/2\Sigma$.*

Lemma 3.6. *Let C be a dividing real irreducible sextic with only non-real nodes. The real structure with $\bar{\alpha} \in \Sigma/2\Sigma$ is the one with property $(*)$ (see § 1.2).*

Proof. Suppose on the contrary that the real structure c with property $(*)$ has $\bar{\alpha} \notin \Sigma/2\Sigma$. Then we can write $\bar{\alpha} = \bar{h} + y$ for some $y \in \Sigma/2\Sigma$. Let $x \in H^2(X; \mathbb{Z})$ be an element such that $c^*(x) = -x$ and $x \cdot h \equiv 1 \pmod{2}$. By Remark 3.3, the element y is a sum of elements of the form $s'_i + s''_i$. We have $x \cdot s'_i = c^*(x) \cdot c^*(s'_i) = x \cdot s''_i$, and hence $x \cdot (s'_i + s''_i) \equiv 0 \pmod{2}$. This leads to $0 \equiv -x^2 = x \cdot c^*(x) \equiv x \cdot \alpha \equiv x \cdot h \equiv 1 \pmod{2}$, a contradiction. \square

Remark 3.7. Lemma 3.6 also holds for curves with empty real part. The real structure with property $(*)$ is the one without real points, and therefore $\bar{\alpha} = 0$.

3.2. Perturbation of nodes. In this subsection we examine how the homological data of a nodal sextic changes (or rather, does not change) when a pair of non-real nodes is perturbed. As a corollary, we show how crossing pairs can be distinguished from non-crossing pairs on the homological level.

The complex case. Let X_0 be a compact complex surface whose only singularity is an ordinary double point at $p \in X_0$. Let $\mathrm{bl}_p : Y_0 \rightarrow X_0$ be the blow-up of X_0 at p . Then the induced homomorphism $\mathrm{bl}_p^* : H^2(X_0; \mathbb{Z}) \rightarrow H^2(Y_0; \mathbb{Z})$ is injective and its image is the orthogonal complement of $s \in H^2(Y_0; \mathbb{Z})$, the class of the exceptional curve of the blow-up. The self-intersection of s is -2 .

Let $f : \mathcal{X} \rightarrow \mathbb{D}$ be a Lefschetz deformation of X_0 over the unit disk $\mathbb{D} \subset \mathbb{C}$. By this we mean that \mathcal{X} is a 3-dimensional complex manifold with $f^{-1}(0) = X_0$, and f is a proper surjective holomorphic map whose differential is non-zero everywhere except at $p \in X_0 \subset \mathcal{X}$.

The space \mathcal{X} retracts by deformation to the special fibre X_0 . Indeed, a deformation retraction $r : \mathcal{X} \rightarrow X_0$ can be obtained by choosing a metric on \mathcal{X} and using the gradient flow of the function $|f|^2$.

Let $X_\varepsilon = f^{-1}(\varepsilon)$ be a general fibre of f , and let $i : X_\varepsilon \hookrightarrow \mathcal{X}$ denote its inclusion. In the following proposition we collect some facts from Picard-Lefschetz theory. A proof of these facts can be found for instance in the book of C. Voisin [19, ch. 3].

Proposition 3.8.

- (1) *The kernel of $i_* : H_2(X_\varepsilon; \mathbb{Z}) \rightarrow H_2(\mathcal{X}; \mathbb{Z})$ is generated by the class of a 2-dimensional sphere in X_ε , called a vanishing cycle. Let $\delta \in H^2(X_\varepsilon; \mathbb{Z})$ be the class Poincaré dual to a vanishing cycle. It is well-defined up to sign and has self-intersection number -2 .*
- (2) *The monodromy action on $H^2(X_\varepsilon; \mathbb{Z})$ is given by the Picard-Lefschetz map:*

$$\begin{aligned} R_\delta : H^2(X_\varepsilon; \mathbb{Z}) &\rightarrow H^2(X_\varepsilon; \mathbb{Z}) \\ x &\mapsto x + (x, \delta) \cdot \delta \end{aligned}$$

- (3) *The map $i^* : H^2(\mathcal{X}; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ is injective and its image consists of the classes invariant under R_δ , i.e., the orthogonal complement of δ .*

Proposition 3.9. *There are two isometries $\psi : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ which make the following diagram commute. One maps s to δ , the other one to $-\delta$.*

$$\begin{array}{ccc} H^2(Y_0; \mathbb{Z}) & \xrightarrow{\psi} & H^2(X_\varepsilon; \mathbb{Z}) \\ \text{bl}_p^* \uparrow & & \uparrow i^* \\ H^2(X_0; \mathbb{Z}) & \xrightarrow{r^*} & H^2(\mathcal{X}; \mathbb{Z}) \end{array}$$

Proof. The composition $i^* \circ r^* \circ (\text{bl}_p^*)^{-1}$ defines an isomorphism between the orthogonal complements $\langle s \rangle^\perp \subset H^2(Y_0; \mathbb{Z})$ and $\langle \delta \rangle^\perp \subset H^2(X_\varepsilon; \mathbb{Z})$. It can be extended to an isomorphism $\psi : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ by gluing it to an isomorphism between $\langle s \rangle$ and $\langle \delta \rangle$. Such an isomorphism must map s either to δ or to $-\delta$. \square

Remark 3.10. This observation can also be obtained by explicitly constructing the two possible resolutions of the deformation $f : \mathcal{X} \rightarrow \mathbb{D}$, cf. M. Atiyah [1].

Remark 3.11. If we perturb several nodes simultaneously, the situation is very similar. Let X_0 be a surface with n nodes at p_1, \dots, p_n , let $s_1, \dots, s_n \in H^2(Y_0; \mathbb{Z})$ be the exceptional classes of the blow-up, and let $\delta_1, \dots, \delta_n \in H^2(X_\varepsilon; \mathbb{Z})$ denote the classes of the corresponding vanishing cycles. If the sublattice generated by s_1, \dots, s_n is primitive in $H^2(Y_0; \mathbb{Z})$, then there are 2^n different isomorphisms ψ making the above diagram commute. By the gluing condition, s_i must be mapped either to δ_i or $-\delta_i$, and the signs can be chosen independently.

The real case. If the surface X_0 is equipped with a real structure $c : X_0 \rightarrow X_0$, then we may choose the Lefschetz deformation to be real, i.e., so that c extends to a real structure $c : \mathcal{X} \rightarrow \mathcal{X}$ which lifts the complex conjugation on \mathbb{D} . In this situation, we take the general fibre X_ε over a real point $\varepsilon \in \mathbb{D}^* \cap \mathbb{R}$, so that c defines a real structure on X_ε . The maps r^*, i^* and bl_p^* above are equivariant with respect to the real structures.

Proposition 3.12 (Perturbation of a real node). *Let X_0 be a real surface with a real node. Choose a real deformation $f : \mathcal{X} \rightarrow \mathbb{D}$, and let $\psi : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ be an isomorphism as in Proposition 3.9. If $\phi_0 : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(Y_0; \mathbb{Z})$ and $\phi_\varepsilon : H^2(X_\varepsilon; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ denote the involutions induced by the respective real structures, then we have*

$$\phi_\varepsilon = \begin{cases} \psi \circ \phi_0 \circ \psi^{-1} & \text{or} \\ \psi \circ \phi_0 \circ \psi^{-1} \circ R_\delta \end{cases}$$

depending on the sign of ε , where R_δ denotes the Picard-Lefschetz map.

Proof. Since all the involved maps are equivariant with respect to the real structures, it follows from the commutative diagram above that ϕ_ε agrees with $\psi \circ \phi_0 \circ \psi^{-1}$ on the orthogonal complement of δ . Therefore ϕ_ε acts on $\langle \delta \rangle$ as ± 1 . It remains to show that this sign really depends on the sign of ε . To see this, consider a path γ connecting ε with $-\varepsilon$ in the upper half plane (see Figure 3) and let $M(\gamma) : H^2(X_\varepsilon; \mathbb{Z}) \rightarrow H^2(X_{-\varepsilon}; \mathbb{Z})$ be the monodromy along γ . Because f is real (that is, $f \circ c = \text{conj} \circ f$), we have $M(\text{conj} \circ \gamma^{-1}) = \phi_\varepsilon \circ M(\gamma^{-1}) \circ \phi_{-\varepsilon}$. Moreover, since the concatenation of the paths γ and $\text{conj} \circ \gamma^{-1}$ makes a full turn around 0, the composition $M(\text{conj} \circ \gamma^{-1}) \circ M(\gamma)$ is the Picard-Lefschetz map R_δ . It follows that $\phi_\varepsilon = M(\gamma)^{-1} \circ \phi_{-\varepsilon} \circ M(\gamma) \circ R_\delta$. Therefore, if ϕ_ε maps the vanishing cycle δ_ε to $a\delta_\varepsilon$, $a \in \{\pm 1\}$, then $\phi_{-\varepsilon}$ maps the vanishing cycle $\delta_{-\varepsilon}$ to $-a\delta_{-\varepsilon}$. \square

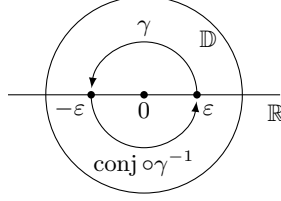


FIGURE 3.

Proposition 3.13 (Perturbation of a pair of non-real nodes). *Let X_0 be a real surface with a pair of non-real nodes. Choose a real deformation $f : \mathcal{X} \rightarrow \mathbb{D}$ of this pair. Then there is an isomorphism $\psi : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ such that $\phi_\varepsilon = \psi \circ \phi_0 \circ \psi^{-1}$.*

Proof. Let $s', s'' \in H^2(Y_0; \mathbb{Z})$ denote the classes of the exceptional curves, and let $\delta', \delta'' \in H^2(X_\varepsilon; \mathbb{Z})$ be the corresponding vanishing cycles. By Proposition 3.9 and Remark 3.11, there is an isomorphism $\psi : H^2(Y_0; \mathbb{Z}) \rightarrow H^2(X_\varepsilon; \mathbb{Z})$ with $\psi(s') = \delta'$, $\psi(s'') = \delta''$, and such that ϕ_ε agrees with $\psi \circ \phi_0 \circ \psi^{-1}$ on the orthogonal complement of $\langle \delta', \delta'' \rangle$. Moreover, $\psi \circ \phi_0 \circ \psi^{-1}$ maps δ' to $-\delta''$, and ϕ_ε must map δ' either to $-\delta''$ or to δ'' . In the former case we are done, and in the latter case it suffices to replace ψ by $R_{\delta'} \circ \psi$. \square

Corollary 3.14. *Let C be a real irreducible sextic whose only singularities are non-real nodes, and let C' be a real non-singular sextic obtained from C by perturbing all the nodes. Let (X, h, c) and (X', h', c') be the weakly polarized real K3 surfaces obtained from C and C' respectively. Then there is an isomorphism $\psi : H^2(X; \mathbb{Z}) \rightarrow H^2(X'; \mathbb{Z})$ such that $\psi(h) = h'$ and $\psi \circ c^* = c'^* \circ \psi$.*

Corollary 3.15. *Let C be a dividing real irreducible sextic whose only singularities are non-real nodes, let (X, h, c) be the associated weakly polarized real K3 surface, let $\alpha \in H^2(X; \mathbb{Z})$ be a characteristic element of c^* and let $\sigma = \{s'_1, s''_1, \dots, s'_m, s''_m\}$ denote its set of nodal classes. Then we have $\alpha \equiv \sum_{i \in I} s'_i + s''_i \pmod{2}$, where $I \subset \{1, \dots, m\}$ is the set of indices of the crossing pairs.*

Proof. By Remark 3.3 and Lemma 3.6, we can write $\alpha \equiv \sum_{i \in I} s'_i + s''_i \pmod{2}$ for some set $I \subset \{1, \dots, m\}$. To see whether a particular $i \in \{1, \dots, m\}$ belongs to I , consider a curve C' obtained from C by perturbing the pair of nodes in question while keeping all the other nodes. Let (X', h', c') be the corresponding weakly polarized real K3 surface, with set of nodal classes σ' . Let $\psi : H^2(X; \mathbb{Z}) \rightarrow H^2(X'; \mathbb{Z})$ be an isomorphism as in Proposition 3.13. A characteristic element for c'^* is given by $\alpha' := \psi(\alpha)$. The i -th pair of non-real nodes is crossing if and only if C' is not dividing. By Proposition 3.1, this is the case if and only if α' is not contained (modulo 2) in the subgroup generated by σ' , which is equivalent to α not being contained (modulo 2) in the subgroup generated by $\psi^{-1}(\sigma') = \sigma \setminus \{s'_i, s''_i\}$. This happens if and only if $i \in I$. \square

4. CLASSIFICATION OF THE HOMOLOGICAL TYPES

4.1. Discriminant groups and discriminant forms. The notions of *discriminant group* and *discriminant form* of a lattice play an important role in the classification of the homological types. In the following we recall the most important definitions; all the necessary details can be found in Nikulin's articles [14] and [15].

A *lattice* is a finitely generated free abelian group L endowed with an integral symmetric bilinear form $b: L \times L \rightarrow \mathbb{Z}$. We usually write $x \cdot y$ instead of $b(x, y)$ and x^2 instead of $b(x, x)$. A lattice L is *even* if x^2 is even for all $x \in L$. The *correlation homomorphism* of a lattice L is the homomorphism $L \rightarrow L^* = \text{Hom}(L, \mathbb{Z})$ defined by $x \mapsto (y \mapsto x \cdot y)$. A lattice is called *non-degenerate* if its correlation homomorphism is injective, and *unimodular* if its correlation homomorphism is bijective. The *discriminant group* of a non-degenerate lattice L is $A_L = L^*/L$, the cokernel of its correlation homomorphism. The discriminant group is a finite abelian group. We may view L^* as a subgroup of $L \otimes \mathbb{Q}$; therefore L^* inherits a \mathbb{Q} -valued bilinear form. This induces a bilinear form $b_L: A_L \times A_L \rightarrow \mathbb{Q}/\mathbb{Z}$, called the *discriminant bilinear form* of L . If L is even, then the squares of elements in A_L are well-defined modulo $2\mathbb{Z}$, i.e. we have a finite quadratic form $q_L: A_L \rightarrow \mathbb{Q}/2\mathbb{Z}$, called the *discriminant (quadratic) form* of L . Note that the discriminant bilinear form can be recovered from the discriminant quadratic form using the identity $b_L(x, y) = \frac{1}{2}(q_L(x + y) - q_L(x) - q_L(y))$.

4.2. Statement of the classification. We fix the number $m \geq 1$ of pairs of non-real nodes. (The case $m = 0$ corresponds to non-singular sextics, for which the homological types are classified by Nikulin [14]. We exclude it here to avoid dealing with certain boundary conditions.) As in the previous sections, we consider a triple (\mathbf{L}, h, σ) , where \mathbf{L} is a fixed K3 lattice, $h \in \mathbf{L}$ is a fixed polarization vector with $h^2 = 2$, and $\sigma \subset L$ is a set of $2m$ pairwise orthogonal roots orthogonal to h , such that the sublattice S generated by h and σ is primitive in \mathbf{L} . By Proposition 1.5, all such triples are isomorphic to each other.

Recall that a *geometric involution* on (\mathbf{L}, h, σ) is an isometric involution $\phi: \mathbf{L} \rightarrow \mathbf{L}$ sending h to $-h$ and σ to $-\sigma$ (as a set), such that the lattice $\mathbf{L}_+ = \{x \in \mathbf{L} \mid \phi(x) = x\}$ has one positive square, and ϕ satisfies condition $(*)$ (see § 1.2). Two such involutions, say ϕ_1 and ϕ_2 , are called *equivalent* if there is an automorphism $\rho: \mathbf{L} \rightarrow \mathbf{L}$ with $\rho(h) = h$ and $\rho(\sigma) = \sigma$ such that $\phi_2 \circ \rho = \rho \circ \phi_1$. Our aim is to classify homological types $(\mathbf{L}, h, \sigma, \phi)$ up to this equivalence.

Convention 4.1. In this section, whenever we speak about a geometric involution ϕ , we assume that it corresponds to sextics without real nodes, i.e. there is no root $s \in \sigma$ with $\phi(s) = -s$. In other words, we use “geometric involution” as a shorthand for “geometric involution corresponding to real irreducible sextics whose only singularities are m pairs of non-real nodes”, and likewise for “homological type”.

Definition 4.2. We define invariants a, t, δ and r for a homological type $(\mathbf{L}, h, \sigma, \phi)$ as follows.

- a : Since ϕ is an involution on a unimodular lattice, the discriminant group of the invariant sublattice \mathbf{L}_+ , which we denote by A_q , is of period 2. Let a be the length of this group. In other words, we have $A_q \cong (\mathbb{Z}/2\mathbb{Z})^a$.
- t : Let t be the number of negative squares of the invariant lattice \mathbf{L}_+ .
- δ : Let $\alpha \in \mathbf{L}$ be a characteristic element of the involution ϕ , and let $\bar{\alpha}$ be its residue in $\mathbf{L}/2\mathbf{L}$ (see § 3.1). Recall that $S/2S$ is naturally embedded in $\mathbf{L}/2\mathbf{L}$. Let $\delta = 0$ if $\bar{\alpha} \in S/2S$ and $\delta = 1$ otherwise.
- r : The invariant r is only defined if $\delta = 0$. If we label the nodal classes $\sigma = \{s'_1, s''_1, \dots, s'_m, s''_m\}$ such that $\phi(s'_i) = -s''_i$ for $i = 1, \dots, m$, then the characteristic element α can be expressed as a sum

$$\alpha \equiv \sum_{i \in I} s'_i + s''_i \pmod{2}$$

for some subset $I \subset \{1, \dots, m\}$, cf. Remark 3.3 and Lemma 3.6. Let r be the cardinality of I .

Note that equivalent homological types have the same invariants.

Theorem 4.3. *Two homological types are equivalent if and only if they have the same invariants (a, t, δ, r) .*

Theorem 4.4. *A homological type $(\mathbf{L}, h, \sigma, \phi)$ with invariants (a, t, δ, r) exists if and only if a, t, δ and r satisfy the following conditions.*

- (i) $a \leq 1 + t \leq 20 - a$.
- (ii) $a \equiv 1 + t \pmod{2}$.
- (iii) $2m \leq a$, with equality only if $\delta = 0$.
- (iv) If $\delta = 0$, then $2r \equiv 1 - t \pmod{4}$.
- (v) If $\delta = 0$ and $a = 1 + t$, then $2r \equiv a - 2 \pmod{8}$.

4.3. Proof of Theorem 4.3.

Definition 4.5 (Nikulin [15]). A *condition on an involution* is a triple (S, θ, G) formed by an even lattice S , an involution $\theta : S \rightarrow S$ and a normal subgroup G of $\{g \in O(S) \mid g \circ \theta = \theta \circ g\}$. A *(unimodular) involution with condition* (S, θ, G) is a triple (L, i, ϕ) formed by a (unimodular) even lattice L , a primitive embedding $i : S \hookrightarrow L$ and an involution ϕ on L such that $\phi \circ i = i \circ \theta$. Two triples (L_1, i_1, ϕ_1) and (L_2, i_2, ϕ_2) are called *isomorphic* if there is an isometry $\rho : L_1 \rightarrow L_2$ such that $\rho \circ \phi_1 = \phi_2 \circ \rho$ and there is $g \in G$ such that $\rho \circ i_1 = i_2 \circ g$.

For a fixed condition (S, θ, G) , Nikulin [15] defines invariants which uniquely determine the genus of involutions with condition (S, θ, G) . (Roughly speaking, two triples (L_1, ϕ_1, i_1) and (L_2, ϕ_2, i_2) have the same *genus* if they are isomorphic over \mathbb{Q} and over the p -adic numbers \mathbb{Z}_p for all primes p .) Moreover he describes all the values these invariants can take, and he gives conditions under which all involutions of the same genus are isomorphic.

The classification of homological types fits into this framework. To see this, we define a condition (\hat{S}, θ, G) as follows. To distinguish abstract lattices and vectors from their images in \mathbf{L} , we put a hat over the corresponding symbols. Let \hat{S} be the lattice $\langle 2 \rangle \oplus \langle -2 \rangle^{2m}$ with fixed orthogonal basis $\{\hat{h}\} \cup \hat{\sigma}$ where $\hat{h}^2 = 2$ and $\hat{\sigma} = \{\hat{s}'_1, \hat{s}''_1, \dots, \hat{s}'_m, \hat{s}''_m\}$. Let the involution $\theta : \hat{S} \rightarrow \hat{S}$ be defined by $\theta(\hat{h}) = -\hat{h}$, $\theta(\hat{s}'_i) = -\hat{s}''_i$ and $\theta(\hat{s}''_i) = -\hat{s}'_i$. Finally, let $G = \{g \in O(\hat{S}) \mid g(\hat{h}) = \hat{h} \text{ and } g(\hat{\sigma}) = \hat{\sigma}\}$. The group G is a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^m \rtimes S_m$, where a permutation $f \in S_m$ takes \hat{s}'_i to $\hat{s}'_{f(i)}$ and \hat{s}''_i to $\hat{s}''_{f(i)}$, and the i -th basis vector $e_i \in (\mathbb{Z}/2\mathbb{Z})^m$ acts by flipping the pair $\{\hat{s}'_i, \hat{s}''_i\}$ and leaving the other basis vectors fixed.

Lemma 4.6. *There is a one-to-one correspondence between geometric involutions ϕ on the fixed triple (\mathbf{L}, h, σ) , considered up to equivalence, and involutions with condition (\hat{S}, θ, G) , say (L, ϕ, i) , where L is isomorphic to \mathbf{L} and such that ϕ is a geometric involution on L , considered up to isomorphism.*

Proof. Given a triple (L, i, ϕ) , by Proposition 1.5 there is always an isometry $\rho : L \rightarrow \mathbf{L}$ such that $\rho \circ i$ maps $\hat{h} \in \hat{S}$ to the fixed vector $h \in \mathbf{L}$ and $\rho \circ i(\hat{\sigma})$ is the fixed set $\sigma \subset \mathbf{L}$. Moreover, isomorphic triples give rise to equivalent involutions. To pass from a geometric involution ϕ on (\mathbf{L}, h, σ) to an involution with condition, one can set $L = \mathbf{L}$ and choose a bijection between $\hat{\sigma}$ and σ mapping pairs $\{\hat{s}'_i, \hat{s}''_i\}$

to pairs of roots exchanged by $-\phi$. Such a bijection always exists and is uniquely determined up to the action of G . \square

In [15, Theorem 1.6.3] Nikulin gives a complete system of invariants for the genus of involutions with condition. In order to prove Theorem 4.3, it suffices to show that the isomorphism classes are determined by the genus, and that the invariants m , a , t , δ and r (see Definition 4.2) determine Nikulin's invariants for the genus.

Uniqueness in the genus. Given a homological type $(\mathbf{L}, h, \sigma, \phi)$ we define the following sublattices of \mathbf{L} .

$$\begin{aligned} \mathbf{L}_+ &= \{x \in \mathbf{L} \mid \phi(x) = x\} & \mathbf{L}_- &= \{x \in \mathbf{L} \mid \phi(x) = -x\} \\ S_+ &= \mathbf{L}_+ \cap S & S_- &= \mathbf{L}_- \cap S \\ K_+ &= \mathbf{L}_+ \cap S^\perp & K_- &= \mathbf{L}_- \cap S^\perp \end{aligned}$$

By [15, Remark 1.6.2], for an isomorphism class of an involution with condition to be unique in its genus it is sufficient that the lattices K_\pm are unique in their genera and that the natural homomorphisms $O(K_\pm) \rightarrow O(k_\pm)$ are surjective.

Proposition 4.7. *If K is an indefinite even lattice whose discriminant form k is of period 4, then K is unique in its genus and the homomorphism $O(K) \rightarrow O(k)$ is surjective.*

Proof. For the case where the length of the discriminant group k is smaller than the rank of the lattice K , see [14, Theorem 1.14.2]. If the length of k is equal to the rank of K and is at least 3, the statement follows from a result of Miranda and Morrison [11, chapter 8, Corollary 7.8]. For lattices of rank 2 with k of period 4, the statement can be verified by hand. The only such lattices are \mathbf{U} , $\langle 2 \rangle \oplus \langle -2 \rangle$, $\mathbf{U}(2)$, $\langle 2 \rangle \oplus \langle -4 \rangle$, $\langle 4 \rangle \oplus \langle -4 \rangle$ and $\mathbf{U}(4)$. \square

Corollary 4.8. *Let $(\mathbf{L}, h, \sigma, \phi)$ be a homological type. Then the lattices K_+ and K_- are unique in their genera, and the natural homomorphisms $O(K_\pm) \rightarrow O(k_\pm)$ are surjective. In particular, the homological type is unique in its genus.*

Proof. Both K_+ and K_- have one positive square and their discriminant groups are of period 4. Therefore they are either indefinite, and we can apply the proposition, or they are of rank 1, and the statement is trivially verified. For the uniqueness in the genus, see [15, Remark 1.6.2]. \square

Invariants for the genus. In the following we define Nikulin's invariants of the genus and show that for involutions with condition (\hat{S}, θ, G) as in Lemma 4.6, they are determined by the invariants m , a , t , δ and r (see Definition 4.2).

Invariants of (\mathbf{L}, ϕ) . Let A_q be the discriminant group of the lattice \mathbf{L}_+ , and let q be the discriminant form defined on it. The invariants characterizing (\mathbf{L}, ϕ) are the rank and signature of \mathbf{L} , the rank and signature of \mathbf{L}_+ and the invariants of q . Since \mathbf{L} is the fixed K3 lattice, its rank and signature are fixed. Since ϕ is a geometric involution, \mathbf{L}_+ has one positive square. Therefore, its rank and signature are determined by t , its number of negative squares. Finally the group A_q is of period 2, and hence q is determined by the length of A_q , which is a , and by the parity of q (cf. [14, Theorem 3.6.2]). The form q is even if and only if the characteristic element $\bar{\alpha} \in \mathbf{L}/2\mathbf{L}$ is zero. This is the case if and only if $\delta = 0$ and $r = 0$.

The groups H_+ and H_- . Let A_{S_+} and A_{S_-} be the discriminant groups of the lattices S_+ and S_- , respectively. Following [15], we define subgroups Γ_\pm and H_\pm of A_{S_\pm} as follows.

$$\begin{aligned}\Gamma_\pm &= \{x \in S_\pm^* \mid \exists y \in S_\mp^* \text{ such that } x + y \in S\} / S_\pm \subset S_\pm^* / S_\pm = A_{S_\pm} \\ H_\pm &= \{x \in S_\pm^* \mid \exists y \in \mathbf{L}_\mp^* \text{ such that } x + y \in \mathbf{L}\} / S_\pm \subset S_\pm^* / S_\pm = A_{S_\pm}\end{aligned}$$

Note that the subgroups Γ_+ and Γ_- depend only on the condition (\hat{S}, θ, G) , whereas the subgroups H_\pm depend a priori on the involution ϕ . However, it turns out that in our case H_+ and H_- do not depend on ϕ . We have $S_+ = \langle p_1, \dots, p_m \rangle$ and $S_- = \langle n_1, \dots, n_m, h \rangle$ where $p_i = s'_i - s''_i$ and $n_i = s'_i + s''_i$. The groups A_{S_\pm} and Γ_\pm are given as follows.

$$\begin{aligned}A_{S_+} &= \langle [\frac{p_1}{4}], \dots, [\frac{p_m}{4}] \rangle && \cong (\mathbb{Z}/4\mathbb{Z})^m \\ A_{S_-} &= \langle [\frac{n_1}{4}], \dots, [\frac{n_m}{4}], [\frac{h}{2}] \rangle && \cong (\mathbb{Z}/4\mathbb{Z})^m \oplus \mathbb{Z}/2\mathbb{Z} \\ \Gamma_+ &= \langle [\frac{p_1}{2}], \dots, [\frac{p_m}{2}] \rangle && \cong (\mathbb{Z}/2\mathbb{Z})^m \subset A_{S_+} \\ \Gamma_- &= \langle [\frac{n_1}{2}], \dots, [\frac{n_m}{2}] \rangle && \cong (\mathbb{Z}/2\mathbb{Z})^m \subset A_{S_-}\end{aligned}$$

The group H_+ contains Γ_+ and is contained in the 2-torsion subgroup of A_{S_+} . Since Γ_+ is the 2-torsion subgroup of A_{S_+} , we have $H_+ = \Gamma_-$. Similarly, H_- contains Γ_- and is contained in the 2-torsion subgroup of A_{S_-} . The group Γ_- is of index two in the 2-torsion subgroup of A_{S_-} , so H_- must be either Γ_- or $\Gamma_- \oplus \langle [\frac{h}{2}] \rangle$. Because we chose the real structure with property $(*)$ (see § 1.2), the vector h does not glue to L_+ , hence we have $[\frac{h}{2}] \notin H_-$ and therefore $H_- = \Gamma_-$.

The finite quadratic form q_r . The lattice S determines an anti-isometry $\gamma : \Gamma_+ \rightarrow \Gamma_-$. We define the lattice $H_+ \oplus_\gamma H_-$ as the gluing of H_+ and H_- along γ . More precisely, let

$$H_+ \oplus_\gamma H_- = (\Gamma_\gamma)_{H_+ \oplus H_-}^\perp / \Gamma_\gamma,$$

where $\Gamma_\gamma \subset \Gamma_+ \oplus \Gamma_-$ is the graph of γ . There is a natural embedding $\gamma_r : H_+ \oplus_\gamma H_- \hookrightarrow q$, inducing a finite quadratic form q_r on $H_+ \oplus_\gamma H_-$. In general (if $\Gamma_+ \oplus_\gamma \Gamma_-$ is strictly contained in $H_+ \oplus_\gamma H_-$) this form q_r depends on the embedding $i : S \hookrightarrow L$ and on the involution ϕ . Since in our case we have $\Gamma_\pm = H_\pm$ however, the form q_r is determined by S and θ .

Invariants of the embedding γ_r . Let $v_q \in A_q$ be the characteristic element of q , i.e. the element for which $(v_q, x) = x^2 \pmod{1}$ for all $x \in A_q$. We have $v_q = [\frac{\alpha}{2}] \in A_q$, where $\alpha \in \mathbf{L}$ is a characteristic element for the involution ϕ . The embedding γ_r is characterized by whether v_q is contained in $\gamma_r(q_r)$, and if this is the case, by the orbit $G \cdot \gamma_r^{-1}(v_q) \subset q_r$. But v_q is contained in $\gamma_r(q_r)$ if and only if $\bar{\alpha} \in \mathbf{L}/2\mathbf{L}$ is contained in $S/2S$, which is by definition if and only if $\delta = 0$. In this case we have $\alpha \equiv \sum_{i \in I} s'_i + s''_i \pmod{2L}$, where the sum ranges over the indices of the crossing pairs. Therefore, $\gamma_r^{-1}(v_q) = \sum_i [\frac{p_i}{2}] \in q_r$ is determined (up to the action of G , which permutes the indices) by r .

This concludes the proof of Theorem 4.3. \square

4.4. Proof of Theorem 4.4.

Proof of Theorem 4.4. By Lemma 4.6, the homological types we consider correspond to involutions with condition (\hat{S}, θ, G) . Therefore we can deduce Theorem 4.4 from Nikulin's existence theorem for involutions with conditions [15, Theorem 1.8.3]. To do this, we need to check that the conditions (i) – (v) of Theorem 4.4 are equivalent to the conditions given in [15, Conditions 1.8.1 and 1.8.2].

For the most part, this verification is straightforward. The only more technical step, which we consider in detail, is the equivalence of condition (v) of Theorem 4.4 with *boundary condition 1* in [15, Conditions 1.8.2], in the case $\delta = 0$. This boundary condition guarantees the existence of a lattice K_+ with prescribed discriminant quadratic form k_+ and index of inertia $(1, t - m)$, using Nikulin's theorem on the existence of an indefinite even lattice with prescribed rank, signature and discriminant form ([14, Theorem 1.10.1]). In our setting, *boundary condition 1* in [15, Conditions 1.8.2] states that

$$(BC1) \quad \text{if } a = 1 + t, \text{ then } 1 - t \equiv 4\varepsilon_{v_+} + c_v \pmod{8}.$$

where c_v and ε_{v_+} are invariants taking values in $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ respectively, whose definition we give below. We need to show that (BC1) is equivalent to condition (v) which states that

$$\text{if } a = 1 + t, \text{ then } 2r \equiv a - 2 \pmod{8}.$$

The equivalence between (BC1) and (v) follows from Lemma 4.11 below. \square

In the following, suppose that $\delta = 0$. Recall from § 4.3 that v_q denotes the characteristic element of the finite quadratic form q .

Definition 4.9. The invariant $c_v \in \mathbb{Z}/8\mathbb{Z}$ is defined such that $\frac{1}{2}c_v = q(v_q) \pmod{2}$ (see [15, p. 113]) and $c_v \in \{-1, 0, 1, 2\} \pmod{8}$ (see [15, p. 118]).

Before defining ε_{v_+} , we introduce two auxiliary finite quadratic forms, γ_+ and η . Let γ_+ be the finite quadratic form

$$\gamma_+ = \begin{cases} \left[\frac{1}{2}\right] \oplus \left[-\frac{1}{2}\right] & \text{if } r \text{ is even,} \\ \left[\frac{1}{2}\right] \oplus \left[\frac{1}{2}\right] & \text{if } r \text{ is odd} \end{cases}$$

(see [15, p. 117, eq. 8.14]), and let x_1 and x_2 be the standard generators of γ_+ . The characteristic element of γ_+ is $v_{\gamma_+} = x_1 + x_2$, and we have $v_{\gamma_+}^2 \equiv r \pmod{2}$.

Let A_{S_+} be the discriminant group of the lattice S_+ , and let s_+ be the finite quadratic form defined on it. Recall that A_{S_+} is generated by the elements $\alpha_i = \left[\frac{s'_i - s''_i}{4}\right]$ for $i \in \{1, \dots, m\}$. Since $q_r \cong \Gamma_- \subset A_{S_-}$, we may view $v = \gamma_r^{-1}(v_q)$ as an element of A_{S_-} (see § 4.3 for the definition of q_r and γ_r). We have $v = \sum_{i \in I} 2\alpha_i$. The finite quadratic form η is obtained by gluing of s_+ and γ_+ , where $v \in s_+$ is identified with $v_{\gamma_+} \in \gamma_+$. More precisely, let $v_\eta = v + v_{\gamma_+} \in s_+ \oplus \gamma_+$. Note that $v_\eta^2 = 0$. Hence we can define the quadratic form

$$\eta = \langle v_\eta \rangle_{(s_+ \oplus \gamma_+)}^\perp / \langle v_\eta \rangle,$$

which is denoted $(q_{S_+^2})_v$ in [15, p. 117, eq. 8.15]. Let $K(\eta_2)$ be a 2-adic lattice of minimal length whose discriminant quadratic form is η . The discriminant of $K(\eta_2)$ is an element of $\mathbb{Z}_2/(\mathbb{Z}_2^*)^2$. There are two such lattices $K(\eta_2)$, and their discriminants differ by a sign.

Definition 4.10. The invariant $\varepsilon_{v_+} \in \mathbb{Z}/2\mathbb{Z}$ is defined by the equation

$$5^{\varepsilon_{v_+}} = \pm \text{discr } K(\eta_2) \in \mathbb{Q}_2/(\mathbb{Q}_2^*)^2.$$

Note that the group $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$ and has representatives $\{\pm 1, \pm 2, \pm 5, \pm 10\}$, see for example J.-P. Serre [18, p. 18].

Lemma 4.11. *If $\delta = 0$, then $4\varepsilon_{v_+} + c_v + 2r \equiv 0 \pmod{8}$.*

Proof. Recall that $v_q = \sum_i [\frac{p_i}{2}] \in q_r$, where the sum ranges over the indices of the crossing pairs. We have $q(v_q) \equiv r \pmod{2}$. Therefore,

$$c_v \equiv \begin{cases} 0 \pmod{8} & \text{if } r \text{ is even,} \\ 2 \pmod{8} & \text{if } r \text{ is odd.} \end{cases}$$

Hence, to prove the lemma, it remains to show that

$$\varepsilon_{v_+} \equiv \begin{cases} 0 \pmod{2} & \text{if } r \equiv 0, 3 \pmod{4}, \\ 1 \pmod{2} & \text{if } r \equiv 1, 2 \pmod{4}. \end{cases}$$

Up to the action of G , we may suppose that $I = \{1, \dots, r\}$. Then a basis of the finite quadratic form η defined above is given by

$$\{\alpha_i + x_1\}_{i=1}^r \cup \{\alpha_i\}_{i=r+1}^m.$$

We define a new basis $\{\beta_i\}_{i=1}^m$ for η by

$$\beta_i = \begin{cases} [2(\alpha_1 + \dots + \alpha_{i-1}) + \alpha_i + x_1] & \text{if } i \in \{1, \dots, r\} \\ [\alpha_i] & \text{if } i \in \{r+1, \dots, m\}. \end{cases}$$

Note that this basis is orthogonal, and we have

$$\beta_i^2 = \begin{cases} -\frac{5}{4} & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ -\frac{1}{4} & \text{otherwise.} \end{cases}$$

Now $K(\eta_2)$ can be defined as the 2-adic lattice with orthogonal basis b_1, \dots, b_m , where

$$b_i^2 = \begin{cases} -5 \cdot 4 & \text{if } i \leq r \text{ and } i \text{ is odd,} \\ -1 \cdot 4 & \text{otherwise.} \end{cases}$$

We calculate the discriminant of $K(\eta_2)$ with respect to this basis. Since the basis is orthogonal, the discriminant is just the product $b_1^2 \cdot b_2^2 \cdot \dots \cdot b_m^2$. The invariant ε_{v_+} is the number of occurrences of the factor 5 in this product, counted modulo 2. This is equal to the number of odd integers between 1 and r , and it is finally easy to check that

$$\varepsilon_{v_+} \equiv \left\lceil \frac{r}{2} \right\rceil \equiv \begin{cases} 0 & \text{if } r \equiv 0, 3 \pmod{4} \\ 1 & \text{if } r \equiv 1, 2 \pmod{4} \end{cases} \pmod{2}. \quad \square$$

5. PROOF OF THEOREM 1 AND THEOREM 2

Proposition 5.1 (Geometrical interpretation of a, t, δ and r). *Let $C \subset \mathbb{CP}^2$ be a real irreducible sextic whose only singularities are m pairs of non-real nodes, and let $(\mathbf{L}, h, \sigma, \phi)$ be its homological type. Then the invariants (a, t, δ, r) have the following geometrical interpretation: if \mathbb{RC} is empty, we have $a = 10$, $t = 9$, $\delta = 0$ and $r = 0$; if \mathbb{RC} is not empty, we have*

$$\begin{aligned} a &= 11 - l, \text{ where } l \text{ is the number of ovals of } C, \\ t &= 9 + \chi(B), \text{ where } B \text{ denotes the non-orientable half of } \mathbb{RP}^2 \setminus \mathbb{RC}, \\ \delta &= \begin{cases} 1 & \text{if } C \text{ is dividing and} \\ 0 & \text{otherwise,} \end{cases} \\ r &\text{ is the number of crossing pairs of } C. \end{aligned}$$

Proof. The interpretations for a and t are well-known for non-singular sextics, cf. [14, Theorem 3.10.6]. If $C \subset \mathbb{CP}^2$ is a real irreducible sextic whose only singularities are m pairs of non-real nodes, consider a real non-singular sextic $C' \subset \mathbb{CP}^2$ obtained from C by perturbing all the nodes. The perturbation does not change the isotopy type of the real part, so the right hand side of the equations does not change when passing from C to C' . If the homological type of C is $(\mathbf{L}, h, \sigma, \phi)$, then the homological type of the perturbed sextic C' is $(\mathbf{L}, h, \emptyset, \phi)$ for a suitable marking (see § 3.2), and thus a and t do not change when passing from C to C' . The statement about δ is Proposition 3.1, and the statement about r is Corollary 3.15. \square

Proof of Theorem 1. Let C_1 and C_2 be two real irreducible sextics whose only singularities are m pairs of non-real nodes. Suppose that C_1 and C_2 have isotopic real parts, are of the same dividing type and, if they are dividing, have the same number of crossing pairs. Then by Proposition 5.1, the invariants m, a, t, δ, r of their homological types are the same. Hence by Theorem 4.3 their homological types are isomorphic and by Proposition 2.1 the curves C_1 and C_2 are rigidly isotopic. \square

Given an isotopy type of a collection of ovals in \mathbb{RP}^2 , a dividing type, and, if the dividing type is I, a number r of crossing pairs, we can define values a, t, δ and r by taking the geometric interpretation in Proposition 5.1 as a definition.

Lemma 5.2. *If an isotopy type, a dividing type and a number of crossing pairs satisfy conditions (1)–(4) given in the introduction, then the corresponding values a, t, δ and r satisfy the conditions (i)–(v) of Theorem 4.4.*

Proof. Properties (i) and (ii) follow from analogous properties for non-singular sextics (see [14, Theorem 3.4.3]) using the existence of a non-singular sextic with the given isotopy type and the required dividing type (1). Using Proposition 5.1, it is straightforward to verify that the properties (iii), (iv) and (v) follow from Harnack's inequality (2) for \tilde{C} , Arnold's congruence (3) and the complex orientation formula for curves without injective pairs (4), respectively. \square

Lemma 5.3. *Different combinations of isotopy type, dividing type and number of crossing pairs, satisfying conditions (1)–(4), lead to different invariants a, t, δ and r .*

Proof. This follows essentially from the fact that, for non-singular real sextics, the rigid isotopy type is determined by a, t and δ (see [14]). \square

Proof of Theorem 2. It is clear that the conditions (1)–(4) of Theorem 2 are necessary for the existence of a sextic with the desired invariants. To show that they are sufficient, suppose we are given an isotopy type of the real part, a dividing type, a number of pairs of nodes, and (if the dividing type is I) a number of crossing pairs, subject to the conditions (1)–(4). By Lemma 5.2, the corresponding values (a, t, δ, r) satisfy the conditions (i)–(v) of Theorem 4.4. Hence by Theorem 4.4, there is a homological type $(\mathbf{L}, h, \sigma, \phi)$ with these invariants. Since the real period space $\Omega_{h, \sigma, \phi}^0$ is not empty and the period map is surjective (see § 1.2), there are sextics with this homological type. Finally, Lemma 5.3 guarantees that such sextics indeed have the desired isotopy type, dividing type and number of crossing pairs. \square

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UNIVERSITÉ DE GENÈVE and INSTITUT DE MATHÉMATIQUES DE JUSSIEU–PARIS RIVE GAUCHE
Current address: Section de mathématiques, Université de Genève, Villa Battelle, Route de Drize 7, 1227 Carouge, Switzerland
E-mail address: johannes.josi@unige.ch